# Chapter 6 Currency Options

## Quiz Questions

### True-False Questions

_______ 1. The only difference between European and American options is that European options are traded only in Europe while American options are traded only in the US.

_______ 2. The buyer of an option has an obligation to purchase in the case of a call, or sell in the case of a put, while the seller of an option has the right to deliver in the case of a call, or take delivery in the case of a put.

_______ 3. A put offers the holder of an asset protection from drops in the underlying asset's value, while a call provides a potential purchaser protection from an increase in the underlying asset's price.

_______ 4. The intrinsic value of a call is its risk-adjusted expected value.

_______ 5. The immediate exercise value of an option is its value alive.

_______ 6. If a call's strike price exceeds the spot rate, the call is in the money.

_______ 7. If an in-the-money put has positive value, its value is based purely on time value.

_______ 8. An European call will always be at least as valuable as a comparable American call.

_______ 9. An option is always at least as valuable as the comparable forward contract.

_______ 10. Put Call Parity implies that puts and calls written at the forward rate will have different values because if the foreign interest rate exceeds the domestic rate the forward rate is at a discount; therefore, the exchange rate is expected to depreciate making the put more valuable.

_______ 11. Speculators disagree with the market's probability distribution function for an asset's value; that is, they sell assets that the market perceives as overvalued and buy assets that the market perceives as undervalued.

Ans. 1. false; 2. false; 3. true; 4. false; 5. false; 6. false; 7. false; 8. false; 9. true; 10. false; 11. false.

### Multiple-Choice Questions

The exercises below assume that the put and the call both have a strike price equal to *X*, a domestic T-bill has a face value equal to *X*, and both a foreign T-bill and forward contract pay off one unit of foreign currency at expiration. All instruments expire on the same date.

Q1. A forward sale can be replicated by:
- (a) selling a put and buying a call.
- (b) selling a foreign T-bill and buying a domestic T-bill.
- (c) buying a put and selling a call.
- (d) both (b) and (c).
- (e) all of the above.

A1. (d).

Q2. A put can be replicated by:
- (a) buying a call and selling foreign currency forward.
- (b) buying a foreign T-bill and selling a call.
- (c) buying a domestic T-bill, selling a foreign T-bill, and buying a call.
- (d) both (a) and (c).

(e) all of the above.

A2. (d).

Q3. A call can be replicated by:
(a) buying foreign currency forward and buying a put.
(b) buying a foreign T-bill and selling a put.
(c) buying a put, selling a domestic T-bill, and buying a foreign T-bill.
(d) all of the above.
(e) none of the above.

A3. (a) & (c).

Use the following table excerpted from *The Wall Street Journal* of Tuesday, March 22, 1994, to answer the questions below.

| Option & underlying | Strike price | Calls—Last | | | Puts—Last | | |
|---|---|---|---|---|---|---|---|
| | | Apr | May | Jun | Apr | May | Jun |
| **31,250 British Pounds-cents per unit.** | | | | | | | |
| 148.61 | 147 1/2 | r | r | r | 0.95 | 1.80 | r |
| 148.61 | 150 | 0.60 | r | 1.85 | r | r | r |
| 148.61 | 155 | 0.07 | r | 0.57 | r | r | r |
| 148.61 | 157 1/2 | 0.03 | r | r | r | r | r |
| **62,500 German Marks-cents per unit.** | | | | | | | |
| 59.04 | 58 | 1.08 | r | r | 0.35 | 0.65 | 0.90 |
| 59.04 | 58 1/2 | 0.79 | r | 1.35 | 0.46 | r | 1.13 |
| 59.04 | 59 | 0.51 | 0.80 | 1.02 | 0.80 | 1.10 | 1.40 |
| 59.04 | 59 1/2 | 0.35 | r | r | r | r | r |
| **6,250,000 Japanese Yen-100ths of a cent per unit.** | | | | | | | |
| 94.18 | 93 | r | r | r | r | r | 1.29 |
| 94.18 | 93 1/2 | r | r | r | 0.72 | r | r |
| 94.18 | 94 | r | r | r | r | 1.41 | 1.68 |
| 94.18 | 94 1/2 | 0.81 | r | r | 1.12 | r | r |

r—not traded. s—no option offered. Last is premium (purchase price).

Q4. What is the last quote for an April call option on GBP with a strike price of 155?
A4. 0.07.

Q5. What is the last quote for a May put option on DEM with a strike price of 58?
A5. 0.65.

Q6. What is the last quote for a June put option on JPY with a strike price of 93 1/2?
A6. The option was not traded on Monday, March 21.

Q7. For the options below, what is the intrinsic value? Is the intrinsic value greater than, less than, or equal to the option premium?
(a) June call on GBP with a strike price of 150.
(b) May put on GBP with a strike price of 147 1/2.
(c) April call on DEM with a strike price of 59.
(d) June put on DEM with a strike price of 59.
(e) May call on JPY with a strike price of 93.
(f) May put on JPY with a strike price of 94.

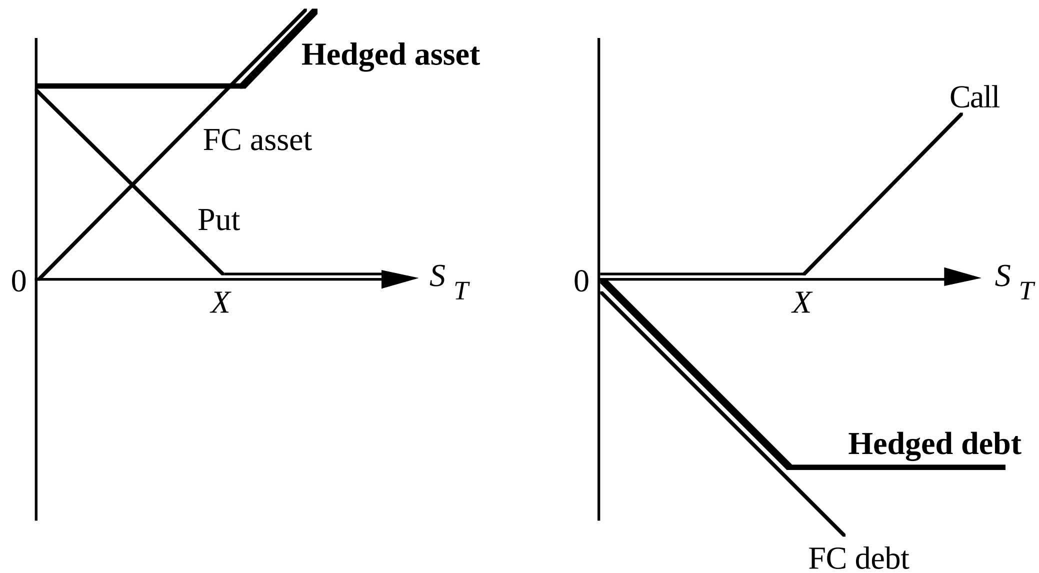
A7. (a) IV = 0 < 1.85 = premium.

(b) IV = 0 < 1.80 = premium.
(c) IV = 0.04 < 0.51 = premium.
(d) IV = 0 < 1.40 = premium.
(e) option not traded.
(f) IV = 0 < 1.41 = premium.

Q8. You hold a foreign exchange asset that you have hedged with a put. Show graphically how the put limits the potential losses created by low exchange rates without eliminating the potential gains from high rates.

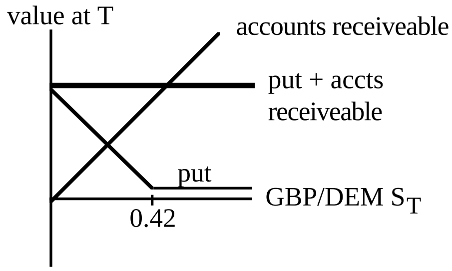
Q9. You have covered a foreign exchange debt using a call. Show graphically how the call limits the potential losses created by high exchange rates, without eliminating the potential gains from low rates.

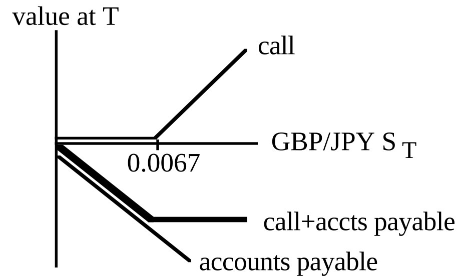
A9. & A10.




Q10. Assume that the contracts discussed below use the GBP as the home currency and the option's expiration date matches the expiration date of the cash flow to be hedged. Illustrate how the exchange rate affects the GBP value of:
(a) a DEM 500,000 accounts receiveable and a purchase of ten puts, each for DEM 50,000, with a strike price of GBP/DEM 0.42.
(b) a JPY 10,000,000 accounts payable and a purchase of ten puts, each for JPY 1,000,000, with a strike price of GBP/JPY 0.0067.

A10. (a) (b)



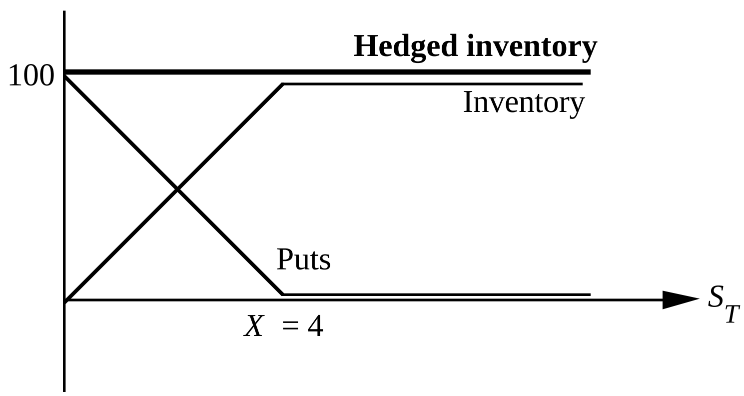




## Exercises

E1. The Danish wool trader in Section 6.5.3 faces potential competition from Australian producers.
   (a) Graphically analyze the value of the trader's inventory as a function of the future spot price.
   (b) Explain why a put on AUD eliminates the dependence of the inventory's value on the exchange rate for DKK/AUD.

A1. (a) For $S_T \geq X = 4$, his stock is worth DKK 100 irrespective of $S_T$ (a flat line); for $S_T < X$, the stock is worth $25 \times S_T$, a ray through the origin with a slope of 25 (see the graph below). This is like a domestic currency bond with face value 100, *minus* a put (that is, with a *shorted* put) struck at $X = 4$.

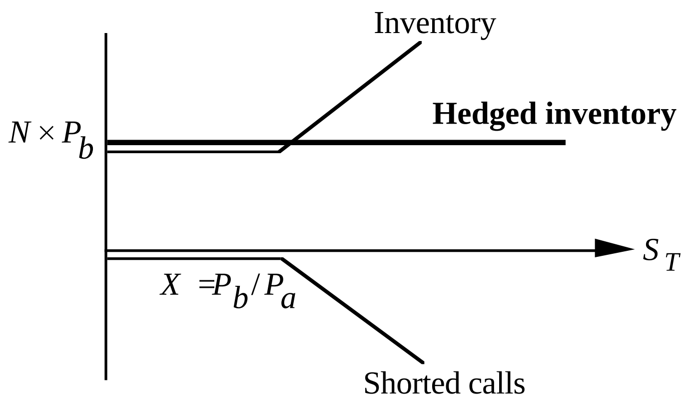
   (b) By buying a put with a strike price $X = 4$, you add a downward-sloping line with an intercept at 100 and with the same (absolute) slope as the inventory line in the domain $S_T < X$. In short, you are just buying back the implicit (shorted) puts.




Note how the price of the puts is the present risk-adjusted expected value of the potential losses created by Australian imports.

E2. The UK firm, Egress Import-Export, Ltd, sells its goods at home for $P_b$ when the value of the ITL is low. As the value of the ITL increases, it starts exporting its goods at the foreign price (net of costs) $P_a$, netting it $P_a \times \tilde{S}_T$.
   (a) Illustrate the value of Egress's goods as a function of the future spot price.
   (b) How can Egress eliminate its exposure to the ITL (that is, sell its potential ITL profits)?

A2. The stock of commodities is worth $N \times P_b$ when $S_T < P_b/P_a$ and $N \times P_a \times S_T$ , otherwise. From the graphical representation, this is like a domestic bond with face value $N \times P_b$, plus $N \times P_a$ calls on foreign exchange with strike price $X = P_b/P_a$. Selling the implicit calls will eliminate exposure. The premium received is the risk-adjusted present value of potential extra gains from exports.




E3. The Luxembourg Plettery Steel Company has a debt of DEM 100,000, which is repayable in 12 months. Plettery's controller Jane Due is having trouble sleeping at night knowing that the debt is unhedged. The current LUF/DEM exchange rate is 20, and *p.a.* interest rates are 21 percent on LUF and 10 percent on DEM. Jane is considering a forward hedge (at $F_{t,T} = 20 \times 1.21/1.10 = 22$); but a friend tells her that he recently bought a call on DEM 100,000 with $X = 20$, and is willing to sell it to her at the historic cost of LUF 1 per DEM or LUF 100,000 for the total contract. What should she do?

A3. The call premium asked by her friend violates the lower bound and therefore is an absolute must. A forward contract at $X = 20$ has a value of (22 - 20)/1.21 = 1.653. Since, unlike a forward contract at $X = 20$, the option cannot have a negative expiration value, and it should be worth more than that.

E4. Assume that the interest rates are 21 percent and 10 percent *p.a.* in Luxembourg and Switzerland, respectively. Consider a call and a put at $X$ = LUF/CHF 21.

(a) What is the lower bound for European options with lives equal to $T - t = 1$ year, six months, three months, one month, when $S_t$ = 18, 20, 22, 24, respectively?

(b) If $S_t$=20, $r_{t,T}$ = 21 percent, $r^*_{t,T}$ = 10 percent, a one-year call with $X$ = LUF/DEM 20 priced at 1 is undervalued. Show that, with this call price, we can buy a synthetic put at a negative price.

A4. For the call, first compute the value of the forward contract at $X = 21$,

$$\frac{S_t}{1 + (T - t) \times 0.10} - \frac{21}{1 + (T - t) \times 0.21}$$

| $S_t$ = | 18 | 20 | 22 | 24 |
|---|---|---|---|---|
| Life = 12 months | -0.99 | 0.83 | 2.64 | 4.46 |
| 6 months | -1.86 | 0.04 | 1.85 | 3.85 |
| 3 months | -2.39 | -0.44 | 1.51 | 3.46 |
| 1 month | -2.79 | -0.80 | 1.18 | 3.16 |

Next, add the bound $C_t > 0$ which, of course, becomes relevant when the value of a comparable forward is negative. Also the option's intrinsic value becomes a minimum value:

| $S_1$ = | 18 | 20 | 22 | 24 |
|---|---|---|---|---|
| *Lower bounds:* | | | | |
| Life = 12 months | 0.00 | 0.83 | 2.64 | 4.46 |
| 6 months | 0.00 | 0.04 | 1.85 | 3.85 |
| 3 months | 0.00 | 0.00 | 1.51 | 3.46 |
| 1 month | 0.00 | 0.00 | 1.18 | 3.16 |
| *Intrinsic value:* | 0.00 | 0.00 | 1.00 | 3.00 |

If at $S_T = 24$ the call is almost a forward purchase contract (so that the call is priced close to its lower bound[1]), its value still exceeds the intrinsic value, and exercise is never optimal. An American call will therefore be priced as if it were European. In this example, the reason is that the foreign interest rate is far below the domestic rate.

For the put, first look at the values of comparable forward sales contracts, and these are the same as the above values of the purchase contracts, except for the sign. The non-negativity bound yields the following floor for our various put prices:

| $S_1$ = | 18 | 20 | 22 | 24 |
|---|---|---|---|---|
| *lower bounds:* | | | | |
| life = 12 months | 0.99 | 0.00 | 0.00 | 0.00 |
| 6 months | 1.86 | 0.00 | 0.00 | 0.00 |
| 3 months | 2.39 | 0.44 | 0.00 | 0.00 |
| 1 month | 2.79 | 0.80 | 0.00 | 0.00 |
| *intrinsic value:* | 3.00 | 1.00 | 0.00 | 0.00 |

If, for instance, at $S_t = 18$ the (European) put is nearly degenerate and therefore trades near the value of a forward sale[2], its value would be below the intrinsic value. Given this, an American put would already have been exercised. Therefore, early exercise of the put does not have an *ex ante* probability equal to zero, and all "alive" American puts must trade above European put prices. If the foreign rate had far exceeded the domestic interest rate, you would not have observed this.

E5. A charitable organization has issued a bond which gives the holder the option to cash in the principal as either USD 10,000 or DEM 20,000. This asset can be viewed as a USD 10,000 bond plus a call on $\text{DEM}_T$ 20,000 at $X$ = USD/DEM 0.5.

(a) Can the bond also be viewed as a DEM bond plus an option.

(b) Explain how the two equivalent views are just an application of Put Call Parity.

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1 With the above figures, that is. In general, the answer could change.

2 Note how the lower bound becomes important not when the put is cheap, but when the lower bound is very high. In other words, a put trading close to its bound is deep in the money, and therefore very valuable.

(c) The strike price, $X$ = USD/DEM, is the natural way of quoting a rate for a US investor. But buying DEM 20,000 at USD/DEM 0.5 is the same as selling USD 10,000 at $X'$ = DEM/USD 2. This way of expressing the transaction makes more sense to a German investor. Restate the conditions of the bonds using this DEM/USD strike price, and make two possible option interpretations from a German investor's point of view.

A5. (a) Yes. The bond may be viewed as a DEM 20,000 bond plus a call on $\text{USD}_T$ 10,000 at $X$ = DEM/USD 2, or as a DEM 20,000 bond plus a put on $\text{DEM}_T$ 20,000 at $X$ = USD/DEM 0.5, or as a USD 10,000 bond with a put on $\text{USD}_T$ 10,000 at $X$ = DEM/USD 2. To check all this, first consider the US point of view, and express all prices in USD. Start with the common sense interpretation of the bond, and then check all of the above combinations where the strike price is in USD/DEM:

| | $S_T$ (USD/DEM) < 0.5 | $S_T$ > 0.5 |
|---|---|---|
| Payoff of bond | 10,000 | $S_T \times 20{,}000$ |
| Interpretation 1: | | |
| USD 10,000 bond | 10,000 | 10,000 |
| + call on DEM at 0.5 | 0 | $(S_T - 0.5) \times 20{,}000$ |
| Total | 10,000 | $S_T \times 20{,}000$ |
| Interpretation 2: | | |
| DEM 20,000 bond | $S_T \times 20{,}000$ | $S_T \times 20{,}000$ |
| + put on DEM at 0.5 | $(0.5 - S_T) \times 20{,}000$ | 0 |
| Total | 10,000 | $S_T \times 20{,}000$ |

(b) Since both views are perfectly and equally correct at time t, it must be that in equilibrium:

$$\frac{10{,}000}{1 + r_{t,T}} + C_t \times 20{,}000 = S_t \frac{20{,}000}{1 + r^*_{t,T}} + P_t \times 20{,}000$$

or, after division by 20,000 and noting that $X$ = 0.5,

$$\frac{X}{1 + r_{t,T}} + C_t = S_t \frac{1}{1 + r^*_{t,T}} + P_t$$

which is Put Call Parity.

(c) To take the German point of view, do the same:

| | $S'_T$ (USD/DEM) < 2 | $S'_T$ > 2 |
|---|---|---|
| Payoff of bond | $S'_T \times 10{,}000$ | 20,000 |
| Interpretation 1: | | |
| USD 10,000 bond | 20,000 | 20,000 |
| + call on DEM at 0.5 | $(S'_T - 2) \times 10{,}000$ | 0 |
| Total | $S'_T \times 10{,}000$ | 20,000 |
| Interpretation 2: | | |
| DEM 20,000 bond | $S'_T \times 10{,}000$ | $S'_T \times 10{,}000$ |
| + put on DEM at 0.5 | 0 | $(2 - S'_T) \times 10{,}000$ |
| Total | $S'_T \times 10{,}000$ | 20,000 |

E6. The software giant, Kludge Systems, has issued a bond that gives the holder the choice between $USD_T$ 10,000, $DEM_T$ 20,000, and $GBP_T$ 5,000. Can Kludge's bond be replicated using simple options?

A6. No. It is tempting to say that you have, for instance, a $USD_T$ 10,000 bond plus a call on $DEM_T$ 20,000 at $X$ = USD/DEM 0.5, plus a call on $GBP_T$ 5,000 at $X$ = USD/GBP 2. But it is possible that, at time $T$, *both* the DEM and the GBP are above their strike prices; if you had two separate calls, you would exercise both of them. In contrast, this bond allows you to exercise only one of the above options. Therefore, you really have a USD 10,000 bond, plus a call on the maximum of (DEM 20,000, GBP 5,000).

E7. You have purchased a zero-coupon FIM bond which gives you the choice between FIM 100,000 at $T_2 = 2$ or FIM 90,000 at $T_1 = 1$.

(a) What options (put and/or call) are implicit in this bond. (Hint: there are two correct descriptions.)

(b) Show that the two equivalent views of this instrument are an application of Put-Call Parity.

A7. You could view this as a:

- Two-year $FIM_2$ 100,000 bond with a put on a one-year $FIM_1$ 90,000 bond with a strike price of $X$ = FIM 90,000; or,
- One-year $FIM_1$ 90,000 bond with a call on the two-year $FIM_2$ 100,000 bond with a strike price of $X$ = FIM 90,000.

Both options are European and expire at $T$. Define $V_T$ as the (FIM) price of the underlying asset, the two-year bond. Verify the claim by looking at the value of the bond at $T = 1$.

| | $V_T \geq 90{,}000$ | $V_T > 90{,}000$ |
|---|---|---|
| Your bond | $V_T$ | 90,000 |
| Interpretation 1: | | |
| Two-year bond | $V_T$ | $V_T$ |
| + call on DEM at 0.5 | 0 | $90{,}000 - V_T$ |
| Total | $V_T$ | 90,000 |
| Interpretation 2: | | |
| One-year bond | 90,000 | 90,000 |
| + call at 90,000 | $V_T$ - 90,000 | 0 |
| Total | $V_T$ | 90,000 |

Since both views are equally correct, it must be that $V_t + P_t = \mathrm{PV}_t\,(90{,}000) + C_t$ where $V_t$ is the value of the underlying asset (the non-dividend paying two-year bond), and 90,000 is the value of the one-year bond. This is Put Call Parity.

E8. The lower bound on a non-degenerate American put (that is, a put where there is still some uncertainty about whether $S_t > X$ or not) is:

$$P_t^{\mathrm{am}} > P_t > \frac{X}{1 + r_{t,T}} - \frac{S_t}{1 + r^*_{t,T}}.$$

Assume that $S_t = 0$ and $r_{t,T} = 0$. Common sense says that you should exercise the put, since the exchange rate cannot fall any farther. Yet the bound $P_t > X$ says that the put should trade above its intrinsic value. Where is the fallacy?

A8. The fallacy is that an exchange rate is literally zero only if it is known never to achieve a positive value again. Then, the option is degenerate, and its value is $P_t = X$ (with equality rather than inequality).

In reality, an exchange rate never quite reaches zero, and it can always fall farther. So if $S_t$ is not quite zero, and if there is no time value lost by postponing exercise, standard logic is still applicable: a forward sale is riskier than a put option, so the put trades above $X - \frac{S_t}{1 + r^*_{t,T}}$ and *a fortiori* above $X - S_t$.

E9. A *cylinder* option on the sale of foreign currency is a contract defined as follows:

- If $\tilde{S}_T < X_1$, you sell foreign exchange at $X_1$, the floor.
- If $\tilde{S}_T > X_2$ where $X_2 > X_1$, you sell at $X_2$, the cap.
- If $X_1 \leq \tilde{S}_T \leq X_2$, you sell at $\tilde{S}_T$.

This contract restricts the uncertainty about the futures sales price to the zone $X_1 \leq \tilde{S}_T \leq X_2$.

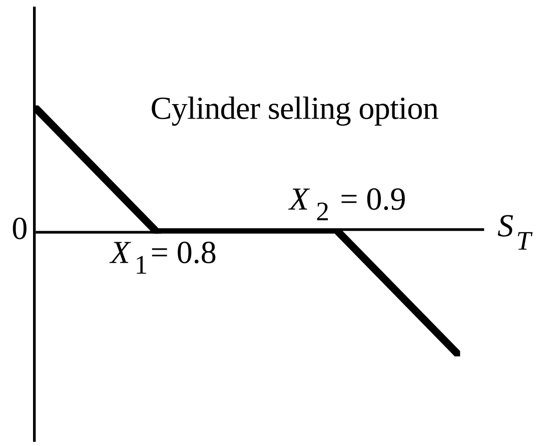
For instance, Barrel Imports has sales contract to sell CAD against USD:

- At $X_1$ = USD/CAD 0.80 if the CAD trades below 0.80.

- At $X_2$ = USD/CAD 0.90 if the CAD trades above 0.90.
- The spot rate if that rate is between 0.80 and 0.90.

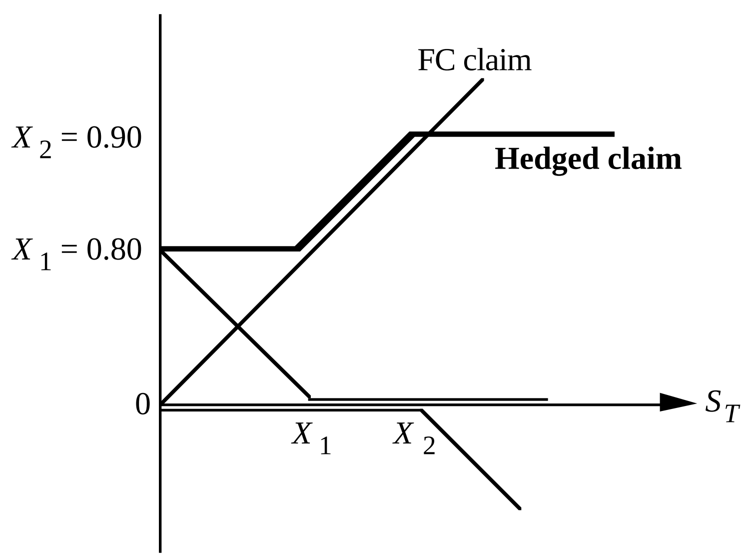
(a) Show the payoff of the contract graphically.
(b) Show that it can be viewed as a combination of European options.
(c) Illustrate the value of a foreign currency claim hedged with such a contract.

A9. (a) If $S_T > X_2$, you lose $S_T - X_2$ since you *have to sell* at the upper bound, which is less than the spot rate. If $S_T < X_1$, you win $X_1 - S_T$, since you *are allowed to sell* at the lower bound, which is above the spot rate. For in-between spot rates, there is neither a gain nor a loss. So the graph looks like the following:




(b) The cylinder option is a combination of a put at $X_1$ (representing *your right* to sell at $X_1$ when $S_T < X_1$) and a *written* call at $X_2$ (representing *the bank's right* to buy from you at $X_2$ when $S_T > X_2$ or of *your obligation* to sell at $X_2$ when $S_T > X_2$). Note that for $X_1 \leq S_T \leq X_2$ both options are worthless, so that the combined value is zero in that domain.

(c) The hedged claim has the following exposure schedule:



Banks usually choose $X_1$ and $X_2$ such that the value of the total package is zero, like for a forward contract, or positive. You could, of course, construct contracts with a negative value (when the value of the call you write exceeds the value of the put, you buy—so that, on balance, the bank pays you something). A cylinder is also called a "collar" or "forward range"; the range $[X_1, X_2]$ is negotiable. The options are European, not American. When $X_1 = X_2 = X$, the forward range collapses to a forward contract with forward price $X$.

## Mind-Expanding Exercises

ME1. Al Say holds an option that gives him the right to "sell" one unit of a DKK T-bill against $X$ units of SEK T-bills, all bills maturing at the same time as the option. Will he exercies this put early?

A1. Use the SEK as the reference currency. For a European option, the expiration value is $(X - S_T)^+$, as all bills are at face value by time $T$. Therefore, this European put has the same value as a European put on DKK struck at SEK/DKK $X$. It follows that unless the put has degenerated into a forward sale,

$$P_{\mathrm{am},t} \geq P_t > \frac{X}{1 + r_{t,T}} - \frac{S_t}{1 + r^*_{t,T}}.$$

The right-hand side also is the early exercise value. Therefore, the put will always be more valuable "alive" than "dead" and it will not be exercised as long as it is nondegenerate.

ME2. Vera Vendible has an option that gives her the right to sell one DKK against SEK $X/(1 + r_{t,T})$ (that is, an option to exchange one DKK against $X$ units of SEK T-bills). Will she exercise early?

A2. No. A European option's expiration value is $(X - S_T)^+$, so the same bound applies for American-type options as in the previous exercise. The *early* exercise value of Vera's option is $\frac{X - S_t}{1 + r_{t,T}}$, which is still above the lowest conceivable market value of the put. Thus, for a put, discounting the strike price suffices to eliminate early exercise.

ME3. What are the implications of Put Call Parity if the probability that $\tilde{S}_T$ exceeds $X$ is zero? Unity?

A3. If $\mathrm{Prob}(S_T > X) = 0$, the call is worthless, and the put will be worth as much as a forward sale. If $\mathrm{Prob}(S_T > X) = 1$, the put is worthless, and the call will be worth as much as a forward purchase.

ME4. You are the holder of a call, expiring at $T$, on a forward purchase contract with the same maturity. When you exercise, you will become the holder of an expiring forward contract with a forward rate equal to $X$. That is, when you exercise, you must buy foreign exchange at the stated forward price $X$. Of course, you will only exercise this option when $\tilde{S}_T > X$. At expiration day, this option's value therefore is $(\tilde{S}_T - X)^+$, just like an ordinary foreign exchange cash option. So if the option is European, it will be priced like a standard option (on cash foreign exchange).

Will options on the forward contract and on cash still be priced equally when the options are American? In other words, can you prove that there will never be any early exercise?

A4. If the call is nondegenerate, that is, the payoff of the option dominates the payoff of the forward contract itself, you can step out by not exercising if $S_T < X$. From[3]

$$(S_T - X)^+ > S_T - X.$$

It follows that, as before, for a European option's price $CF_t$ (Call on a Forward),

$$CF_t > [\text{PV of forward at } X] = \frac{S_t}{1 + r^*_{t,T}} - \frac{X}{1 + r_{t,T}}.$$

The early exercise value of the American option is just the right-hand side. When you exercise before $T$, you're the holder of a forward contract with the value shown on the right-hand side. Since a European call trades above the early exercise value, an American call will do so also, and it will therefore never be exercised early.

ME5. You have the option to buy, at time $T$, a forward purchase contract. The underlying contract matures at $T_2 > T$ (in this case, the forward contract is not yet expiring at $T$). If you exercise, you acquire an asset with a market value equal to $(F_{T,T_2} - X)/(1 + r_{T,T_2})$. Future interest rates are known.

(a) How would you price the European option relative to a European cash foreign exchange option.

(b) Is there any early exercise with American options? When? When not?

A5. (a) You will exercise only when $\frac{F_{T,T_2} - X}{1 + r_{T,T_2}} > 0$, so

$$CF\text{T} = \frac{(F_{T,T_2} - X)+}{1 + r_{T,T_2}}$$

$$= (S_T \frac{1 + r_{T,T_2}}{1 + r^*_{T,T_2}} - X)^+ \times \frac{1}{1 + r_{T,T_2}}$$

$$= \frac{1}{1 + r^*_{T,T_2}}) \;\; (S_T - X')^+ \quad \text{where } X' = X\frac{1 + r^*_{T,T_2}}{1 + r_{T,T_2}}.$$

So the payoff is the same as the payoff on $\frac{1}{1 + r^*_{T,T_2}}$ European options on cash foreign exchange, with strike price $X'$ (not $X$!). The current price of a European option on this forward purchase must therefore be the same as the current value of $\frac{1}{1 + r^*_{T,T_2}}$ European options on cash foreign exchange with strike price $X'$.

(b) From the above, then (where the bracketed term is the lower bound on a unit cash option):

---

[3] The notation "≥" means "greater than or equal to", *with a strictly positive probability of having greater than*. The notation therefore contains the nondegeneracy of the option.

$$CF_t > \frac{1}{1 + r^*_{T,T_2}} \left[ \frac{S_t}{1 + r^*_{t,T}} - \frac{X'}{1 + r_{t,T}} \right] = \text{Early exercise value.}$$

Plugging in $X' = X\frac{1 + r^*_{T,T2}}{1 + r_{T,T2}}$ and noting that, under certainty about interest rates, $(1 + r^*_{t,T})$ $(1 + {}^*_{T,T2}) = (1 + r^*_{t,T2})$ and similarly for $r$, we get:

$$CF_t > \frac{S_t}{1 + r^*_{t,T2}} - \frac{X}{1 + r_{t,T2}} = \text{Early exercise value.}$$

It follows that there will never be early exercise, and that American options of this type will be priced as if they were European.

The intuition behind the two preceding results is that, since a forward contract has unlimited liability and hence may have a negative value at $T$, you should wait until you know for sure that the contract's time $T$ value will be positive. And you will be sure on day $T$ only.

You may be tempted to extrapolate this conclusion to futures contracts, too. That's wrong, as Exercises 12 and 14 show.

ME6. Suppose that your option is an option on a futures contract. Upon exercising the call at time $T$, you acquires a futures purchase contract maturing at $T_2 > T$ with value $(f_{T,T2} - X)$. Future interest rates are known.

(a) How do you price the European option relative to a European cash foreign exchange option.

(b) Is there any problem with the early exercise of American options. When? When not?

A6. (a) You will exercise only when $(f_{T,T2} - X) > 0$, so

$$Cf_T = (f_{T,T2} - X)+ \quad [4]$$

$$= \left(S_T \frac{1 + r_{T,T2}}{1 + r^*_{T,T2}}\right) - X)+$$

$$= \frac{1 + r_{T,T2}}{1 + r^*_{T,T2}} (S_T - X'')+ \quad \text{where } X'' = X\frac{1 + r^*_{T,T2}}{1 + r_{T,T2}}$$

So the payoff is the same as the payoff on $k = (1 + r_{T,T2})/(1 + r^*_{T,T2})$ European options on cash foriegn exchange struck at $X''$ (not $X$!). The current price of a European option on this futures purchase must therefore be the same as the current value of $(1 + r_{T,T2})/(1 + r^*_{T,T2})$ European options on cash foreign exchange, with strike price $X''$.

(b) From the above, then (where the bracketed term is the lower bound on a unit cash option):

---

4 Note again the lack of discounting as compared to the forward contract.

$$Cf_t \;>\; \frac{1 + r_{T,T2}}{1 + r^*_{T,T2}} \left[ \frac{St}{1 + r^*_{t,T}} - \frac{X''}{1 + r_{t,T}} \right]$$

Plugging in $X'' = X\frac{1 + r^*_{T,T2}}{1 + r_{T,T2}}$ and noting that, under certainty about interest rates, $(1 + r^*_{t,T})\,(1 + r^*_{T,T2}) = (1 + r^*_{T,T2})$ and similarly for $r$, you get

$$Cf_t \;>\; (1 + r_{T,T2}) \left[ \frac{S_t}{1 + r^*_{t,T2}}) - \frac{X}{1 + r_{t,T2}} \right] = \text{Early exercise value.}$$

Alternatively: since the payoff at $T$ is $(1 + rT,T2)$ times higher than for the previous option, the current European option's lower bound must be $(1 + r_{T,T2})$ times higher too. But

$$(1 + r_{T,T2}) \;=\; \frac{1 + r_{t,T2}}{1 + r_{t,T}}$$

which allows us to bring out the current futures price:

$$Cf_t \;>\; \frac{1 + r_{t,T2}}{1 + r_{t,T}} \left[ \frac{S_t}{1 + r^*_{t,T2}} - \frac{X}{1 + r_{t,T2}} \right]$$

or

$$Cf_{t\ j} > \frac{1}{1 + r_{t,T}} \left[ St\frac{1 + r_{t,T2}}{1 + r^*_{t,T2}} - X\frac{1 + r_{t,T2}}{1 + r_{t,T2}} \right]$$

$$> \frac{1}{1 + r_{t,T}} \left[ f_{t,T2} - X \right].$$

That is, the lower bound is below the early exercise value, $f_{t,t2} - X$. It follows that there may be early exercise, notably when the option is deep in the money; and that American options of this type will be priced above European options on futures. The only exception is when $r = 0$ (or very low).

Why is this conclusion different from the one derived for an option on a forward? The reason is that the early exercise value of an option on a futures contract involves no discounting, whereas for a forward contract the market value is computed with discounting. That is, in the case of the futures option, you get the time value of money for free. This extra inducement can be sufficient to invalidate the intuitive argument presented for options on forwards.

ME7. Imagine an American call on one DEM against ATS, except that the payment is not ATS $X$ (in cash), but an ATS T-bill with face value $X$ maturing at time $T$. Equivalently, imagine a call on DEM where the payment is ATS $X$ payable at the option's expiration time $T$, even if the option is exercised early. Will such an option be exercised early.

A7. If the option is European, its expiration value is $(S_T\text{-}X)+$, as for the standard call. To see this, note that, at $T$, the T-bill is worth LUF $X$, so that delivering the T-bill is indistinguishable from paying $X$ in cash. Therefore, the option's current price would be the same as that for a standard call. But then its lower bound must be the same, too:

$$C_t > \frac{S_t}{1 + r^*_{t,T}} - \frac{X}{1 + r_{t,T}}.$$

The early exercise value equals the value $S_T$ of the asset bought, minus the cost of the LUF T-bill to be delivered:

$$S_t - \frac{X}{1 + r_{t,T}}.$$

Note that your "payment" now is $X$ discounted (that is, the current value of the T-bill you have to surrender), not $X$. Because, in general, RF is positive, the lower bound is below the early exercise value. In other words, when the option is deep in the money and trades at its lower bound, a European call may trade below its intrinsic value. Therefore, an American option should be worth more than a European option.

ME8. Do the same as in the previous question with an option to acquire a DEM T-bill with face value 1 maturing at time $T$, against the delivery of a LUF T-bill with face value $X$ at $T$. Or, equivalently: will there be any early exercise if both the payment (ATS $X$) and the delivery (DEM 1) are at $T$ even if the decision to exercise is taken before t?

A8. The answer is the same as for question 8, except that now the intrinsic value equals the lower bound. So there cannot be any early exercise.

# Chapter 7 Pricing Currency Options Using the Binomial Model

## Quiz Questions

### True-False Questions

_______ 1. An option's exposure is the sensitivity of a change in the price of the underlying asset to a change in the option's price.

_______ 2. The binomial model uses the risk-adjusted probability $q$ as the certainty equivalent for the unknown (true) probability $p$.

_______ 3. The factor u is the risk-adjusted probability of an upward change in the exchange rate.

_______ 4. Dynamic hedging assumes that at any discrete moment investors can readjust their portfolio holdings.

_______ 5. The delta or exposure of an option is constant.

_______ 6. The delta or hedge ratio is the number of calls one needs in order to replicate one unit of foreign currency.

_______ 7. The probability $\pi$ is a cumulative probability while $(\Sigma_{j=a}^{n}) \left(\binom{n}{j}\right) q^j (1\text{-}q)^{n\text{-}j}$ is a simple probability.

_______ 8. The value of an American option should always be greater than or equal to its intrinsic value.

Ans: 1. false; 2. true; 3. false; 4. true; 5. false; 6. false; 7. false; 8. true.

### Multiple-Choice Questions

Q1. The replication approach to valuing a call option means:
- (a) the payoffs of the call and its underlying asset are always identical.
- (b) buying forward a number of units of the underlying asset such that the payoffs of the option and the forward purchase are identical.
- (c) buying forward a number of units of the underlying asset such that the payoffs of the option and the forward purchase are identical up to a known amount, which is then replicated in the money market.
- (d) selling the call and buying forward a number of units of the underlying asset such that the payoffs are equal to zero.

A1. (c).

Q2. The forward hedging approach to valuing a call option means:
- (a) buying the call and selling forward a number of units of the underlying asset such that the payoffs are equal to the value of a domestic T-bill.
- (b) buying forward a number of units of the underlying asset such that the payoffs of the option and the forward purchase are identical.
- (c) buying the call and selling forward a number of units of the underlying asset such that the payoffs are identical.
- (d) selling the call and buying forward a number of units of the underlying asset such that the payoffs are equal to zero.

A2. (a).

Q3. To compute the certainty equivalent of the future payoff you need:
(a) the true probability $p$.
(b) the risk-adjusted probability $q$.
(c) the expected probability E($p$).
(d) the implied probability of $p$.

A3. (b).

Q4. As the number of periods in the binomial model increases:
(a) the resulting probability distribution of the future spot rate becomes bell shaped.
(b) the resulting probability distribution of the call price becomes bell shaped.
(c) the greater the likelihood that the exchange rate will become negative because of constant additive price changes.
(d) the risk-adjusted expected probability $q$ decreases.

A4. (a).

## Additional Quiz Questions

Q1. If $S_0$ = 100 and the spot rate can increase by 6 percent or decrease by 3 percent, what is the spot rate at node:
(a) (3,3)?
(b) (3,2)?
(c) (3,1)?
(d) (3,0)?

A1. (a) 119.1016 ; (b) 108.9892 ; (c) 99.7354 ; (d) 91.2673.

Q2. Suppose that the current USD/DEM spot rate is 0.6, the effective risk-free rates of return are $r_{0,1}$ = 6 percent and $r^*_{0,1}$ = 8 percent, and the spot rate will either increase to 0.62 or decrease to 0.57 at time 1.
(a) What is the risk-adjusted probability of a spot rate increase? Decrease?
(b) What is the risk-adjusted expected value of European call with a strike price of 0.59?
(c) What is the time-0 value of the call?
(d) What is the factor $u$ ($d$) by which the spot rate increases? Decreases?
(e) What is the option's exposure?
(f) Would the option's value change if it were American?

A2. (a) 0.3778; 0.6222.
(b) 0.01133.
(c) 0.01069.
(d) 1.0333, 0.95.
(e) 0.6000.
(f) No, because the option's value (0.01069) exceeds its intrinsic value (0.01).

Q3. Repeat the question above but for a put with the same strike price, instead of a call.

A3. (a) 0.3778; 0.6222.
(b) 0.01244.
(c) 0.01174.
(d) 1.0333, 0.95.
(e) -0.4.

(f) No, because the option's value (0.01174) exceeds its intrinsic value (0).

## Exercises

E1. In the one-period example of Section 7.1, how could you make risk-free money if the call were valued at 10 rather than at 19.05?

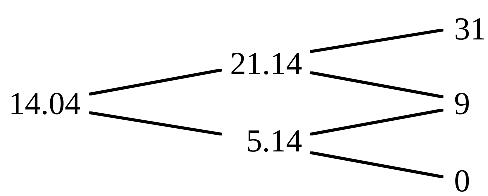
A2. Buy one call using borrowed money (ITL 10), and hedge the call by selling forward DEM 1/3. The hedged call pays off ITL 20 without risk (see Section 7.1), which leaves you a risk-free income of 20 - 10 × 1.05 = 9.5 after paying back the loan. The present value of this future arbitrage income is 9.5/1.05 = 9.05, which is the amount of mispricing on the call relative to the other data.

E2. In the same example, how would you change your answer if you discovered that the probability of "up" were 0.1, so that the exchange rate looked grossly overvalued?

A2. You might be unwilling to buy the stock, or you might be even tempted to sell it short, but the analysis in the preceding exercise remains correct. Given the interest rates and a spot price of ITL 1,000, any call price different from 19.05 will produce arbitrage opportunities.

E3. For the two-period call example in Section 7.7:
(a) Show the tree of European call values if $X$ = 90.
(b) Compare this with the call's intrinsic values in each node.
(c) Check whether there would be a chance of early exercise if the option were American.

A3.




The European call's price always exceeds its intrinsic value due to the fact that the foreign risk-free rate, 0.01, is small compared to the domestic rate, 0.05. If the foreign rate had been zero, you would not have needed the computations to verify this result. *A priori*, you would be sure that $C_t > (S_t - X)^+$.

E4. Consider a one-period call option on the British pound. Suppose that the current exchange rate is USD/GBP 2, the exercise price is USD/GBP 1.9, the one-period risk-free rate on the USD is 5 percent, and the one-period risk-free rate on the GBP is 10 percent. Suppose that the spot rate can either go up by a factor of 1.1 (to USD/GBP 2.2) or down by 0.9 (to USD/GBP 1.8).
(a) Write down the two equations that show how one can replicate the cash flow from the option by investing in the foreign currency and borrowing domestically. What is the value of the call option, using the replication approach?
(b) Compute the risk-neutral probabilities and use these to value the above call option.

A4. (a) $F_{0,1} = 2\,\frac{1.05}{1.10} = 1.9091$ $\quad\quad q = \frac{0.3\ -\ 0}{2.2\ -\ 1.8}.$

(a1) $S_{1u}$: exposure $[S_{1u} - F_{0,1}]$ + $\text{deposit}_{0,1} = \text{call}_{1,u}$.

(a2) $S_{1d}$: exposure $[S_{1d} - F_{0,1}]$ + $\text{deposit}_{0,0}$ = $\text{call}_{1,d}$.

(a1) $S_{1u}$: 0.75 [2.2 - 1.9091] + 0.0818 = 0.3.
(a2) $S_{1d}$: 0.75 [1.8 - 1.9091] + 0.0818 = 0.3.

The present value of the call equals 0.0818/1.05 = 0.0779.

(b) $q = \dfrac{\dfrac{1.05}{1.10} - 0.9}{1.10 - 0.9} = 0.272727.$

Present value of the call equals $\dfrac{0.3 \times 0.273 + 0 \times 0.7273}{1.05} = \dfrac{0.0818}{1.05} = 0.0779.$

E5. Suppose that the current spot rate is $S_0$ = USD/GBP 2 and the one-period interest rates today are $r$ = 5% and $r^*$= 10%. Also, you are given that in the next period the spot rate will either be USD/GBP 2.2 or USD/GBP 1.8.

(a) What is the value today of a one-period put option on the GBP that has a strike of USD/GBP 1.9?

(b) Suppose that you already hold this put option. If you wish to hedge the payoff from the put, so that the net payoff of your portfolio is independent of the exchange rate, how many additional units of the spot should you buy/sell?

A5. (a) $$P_0 = \frac{(1 - 0.272727) \times 0.1}{1.05} = 0.069264.$$

(b) Exposure = $\dfrac{0 - 0.1}{2.2 - 1.8} = 0.25.$

- Forward hedge: sell GBP 0.25 (forward).
- Spot hedge: borrow GBP $0.25 \times \dfrac{1}{1.1}$ = GBP 0.227272 units .

E6. In this exercise, we numerically verify that the probabilities derived for European calls also work for other contracts, by (i) valuing the contracts starting from the value of a call, and (ii) by checking whether a risk-adjusted probability evaluation provides the same answer.

Consider the example used in Section 7.4. The data used were u = 1.1, d = 0.9, $(1 + r) = 1.05$, $(1 + r^*) = 1.0294118$, $S_0 = 100$; for our call, $X = 95$. The tree, including the (risk-adjusted) probabilities for time 2 follows. Ignore the columns added to the right, initially.

| DEM T-bill | | | *Prob* | Call at 95 | Put at 95 | Forward at 95 |
|---|---|---|---|---|---|---|
| | | 121 | *0.36* | 26 | 0 | 26 |
| | 110 | | | | | |
| 100 | | 99 | *0.48* | 4 | 0 | 4 |
| | 90 | | | | | |
| | | 81 | *0.16* | 0 | 14 | -14 |

(a) Compute the call value using the binomial model.

(b) Compute the two-period forward rate directly (using IRP), and indirectly (using our risk-adjusted probabilities).

(c) Compute the present value of an "old" forward purchase struck at $F_{t,2}$ = 95 directly (using the formula in Chapter 3), and indirectly (using $q$).

(d) Value a European put with $X$ = 95 directly (using Put Call Parity), and indirectly (using $q$).

A6. (a) $C_0 = \dfrac{(26 \times 0.36) + (4 \times 0.48) + (0 \times 0.16)}{1.05^2} = 10.23.$

(b) IRP says that $F_{0,2} = 100 \dfrac{1.05^2}{1.0294118^2} = 104.04$, while the "mean" computed with the risk-adjusted probabilities is:

$$\text{CEQ}_0(\tilde{S}_2 = F_{0,2} = (121 \times 0.36) + (99 \times 0.48) + (81 \times 0.16) = 104.04.$$

(c) Directly, the outcome is the discounted value of the difference between the current forward rate 104.04 and the contract's delivery price 95: PV = (1102.5 - 890)/1.052 = 192.7. Indirectly, you can find the same number:

$$\text{PV} = \frac{(26 \times 0.36) + (4 \times 0.48) - (14 \times 0.16)}{1.05^2} = 8.20.$$

(As a parenthesis, compare this computation with the above calculation for the call; clearly, the call is worth more because it avoids the forward contract's negative outcome in node (2,0), to wit, -4. This is a concept that you should clearly understand by now.)

(d) From Put Call Parity, the put should be priced as $C_0$ minus the value of a forward purchase at 95, that is, as:

$$P_0 = 10.23 - 8.20 = 2.03.$$

The risk-adjusted probabilities approach again produces the same value:

$$P_0 = (14 \times 0.16)/1.05^2 = 2.03.$$

(The example also stresses the relation between puts, calls, and forward contracts at $X$ = 95. The call is worth more than the forward purchase because the negative outflow, minus 14 × 0.16 in node (2,0), is missing (compare the payoff columns). This is precisely the term captured by the put, as can be seen from the column showing the put's payoff. So a call minus a put is equivalent to the forward purchase—in terms of expiration values as well as of risk-adjusted present values. Again, this merely illustrates something we knew all along.).

E7. Consider a 4-month call option on the British pound. Suppose that the current exchange rate is USD/GBP 1.6, the exercise price is USD/GBP 1.6, the risk-free rate on the USD is 8 percent *p.a.*, the risk-free rate on GBP is 11 percent *p.a.*, and the volatility of the spot rate (and the forward rate) is 10 percent. Using the results in Appendix 7A, translate the volatility into an up and down factor ($u$ and $d$). Then solve the following problems:

(a) What is the value that you would be willing to pay for this American call option if you used the one-period binomial approach to value it?

(b) What would you be willing to pay for this option if the volatility was 14.1 percent?

A7. (a) USD = home currency

$$u = e^{\sigma\sqrt{h}} = e^{0.11\sqrt{4/12}} = 1.06557; \;\; d = \frac{1}{u} = 0.938466.$$

$$q = \frac{\dfrac{1 + (0.08 \times \frac{4}{12})}{1 + (0.11 \times \frac{4}{12})} - 0.938}{1.066 - 0.938} = 0.4082.$$

The exchange rate in each node at time 1 is shown below:

$$\begin{array}{lll} & 1.7049 \longrightarrow & C_{1,u} = 0.1049 \\ 1.6 & & \\ & 1.5015 \longrightarrow & C_{1,d} = 0 \end{array}$$

The value of a European call at time 0 is: $\dfrac{0.1049 \times 0.4082}{1.02667} = 0.0417118$. Since the intrinsic value of the call is zero, the value of the American call is also 0.0417118.

(b) $u = e^{\sigma\sqrt{h}} = e^{0.141\sqrt{4/12}} = 1.0848117$; $d = \dfrac{1}{u} = 0.921819$.

Because the intrinsic value of the call at time 0 is 0, the value of the American call, equals:

$$C_0^{am} = \frac{0.135699 \times 0}{1.02667} = 0.0555761.$$

E8. Suppose that the spot rate is USD/CAD 0.75 and the volatility of this exchange rate is 4 percent *p.a*. The risk-free rate in the US is 7 percent *p.a*. and in Canada it is 9 percent *p.a*. Suppose that the exercise price is CAD 0.75 and the American put option matures in nine months.
(a) Find the value of this option using the one-period binomial approach.
(b) Find the value of this option using the two-period binomial approach.

A8. (a) Approximately \$0.0285.
(b) Approximately \$0.0430.

E9. A foreign currency put option is equivalent to a position in the foreign currency T-bill and a certain amount of borrowing/lending of the home currency. Is your replicating position in the foreign currency T-bill long or short? Why? Do you borrow or lend the home currency? Why?

A9. You borrow the foreign currency to offset the negative exposure of the put. You must invest in the local currency (otherwise net investment is negative).

E10. Show that $\text{CEQ}_t(\tilde{F}_{T1,T2}) = F_{t,T2}$.

A10. This exercise uses the telescope property of expectations: the expectation of the conditional expectation of *X* is just the expectation of *X*. Formally: in a risk-neutral world, you would have $F_{T1,T2} = \mathrm{E}_{T1}(S_{T2})$ where $\mathrm{E}_{T1}(.)$ is an expectation conditional on time-$T_1$ information. Taking expectations at time $t < T_1$, then,

$$\mathrm{E}_t(F_{T1,T2}) = \mathrm{E}_t(\mathrm{E}_{T1}(S_{T2})) = \mathrm{E}_t(S_{T2}) = F_{t,T2}.$$

That is, in a risk-neutral world, forward rates are a martingale (a randomly continuously compounded accumulating process without drift). In a risk-avert world, the above results remain valid if the expectations are computed over a risk-adjusted distribution, that is, when E(.) is replaced by CEQ(.).

E11. What happens to the value of an option when both $S_0$ and $X$ change by the same factor, holding $u$, $d$, $r$, and $r^*$ constant?

A11. $q = \dfrac{\frac{1 + r_{t,T}}{1 \; r^*_{t,T}} - d}{q - d}$ is unaffected, and the payoffs $(S - X)_+$ or $(X - S)_+$ change by the same factor as $S$ and $X$. Thus, the CEQ and the current price also change by the same factor.

## Mind-Expanding Exercises

ME1. Below, we reproduce the formula for a European call on foreign exchange, with as its arguments the characteristics ($S_0$, $u$, $d$) for the exchange rate process:

$$C_0 = \frac{S_0}{(1 + r^*)^n} \sum_{j=a}^{n} \binom{n}{j} \left(qu \frac{1 + r^*}{1 + r}\right)^j \left((1 - q)\, d \frac{1 + r^*}{1 + r}\right)^{n - j}$$

$$- \frac{X}{(1 + r)^n} \sum_{j=a}^{n} \binom{n}{j} q^j \, (1 - q)^{n - j}.$$

As argued in Appendix 7B, a European call on a foreign T-bill (maturing at the same time as the call itself) will have the same final payoff (and, therefore, also the same current value) as the standard currency option. Suppose, however, that you want to obtain a formula that has as its arguments the characteristics of the T-bill process ($V_0$, $u'$, $d'$, as defined in Appendix 7B). How will the formula look?

A1. $V_0 = S_0\,(1 + r^*)^{-n}$ and $u = u'\,(1 + r^*)$,

so,

$$C_0 = V_0 \sum_{j = a}^{n} \binom{n}{j} (qu'/(1 + r)^n)^j \;\; ((1 - q)d'/1 + r)^{n - j}$$

$$- \frac{X}{(1 + r)^n} \sum_{j = a}^{n} \binom{n}{j} q^j \;\; (1 - q)^{n - j}.$$

Finally, set $A' = qu'/(1 + r)^n$, $1 - A' = (1 - q)d'/(1 + r)^n$.

ME2. Below, we reproduce the formula for a European call on foreign exchange with strike price $X''$, with as its arguments the characteristics ($S_0$, $u$, $d$) for the exchange rate process:

$$C_0 = \frac{S_0}{(1 + r^*)^n} \sum_{j=a}^{n} \binom{n}{j} \left(qu \frac{1 + r^*}{1 + r}\right)^j \left((1 - q)\, d \frac{1 + r^*}{1 + r}\right)^{n - j}$$

$$- \frac{X''}{r^n} \sum_{j=a}^{n} \binom{n}{j} q^j (1 - q)^{n - j}.$$

As argued in Appendix B, a European call on a futures contract with strike price $X''$ that matures at the same time as the call itself, will have the same final payoff and, therefore, also the same current value as the standard currency option. Suppose, however, that you want to obtain a formula that has as its arguments the characteristics of the futures process ($f_0$, $u''$, $d''$, as defined in Appendix B). How will the formula look?

A2. $f_0 = S_0 \left(\frac{1 + r}{1 + r^*}\right)^{n + K}$, where $K$ is the remaining life of the futures contract when the option expires. Write this as $f_0 = S_0 \left(\frac{1 + r}{1 + r^*}\right)^n / k$, with $k = \left(\frac{1 + r}{1 + r^*}\right)^{N - n}$. Divide by $(1 + r^*)^n$ up front and multiply by $(1 + r^*)^j (1 + r^*)^{n - j}$ within the first summation; finally use $u'' = u$\f(1 + $r$,1 + $r^*$) and $d'' = d \frac{1 + r}{1 + r^*}$ so as to obtain:

$$C_0 = \frac{f_0}{k} \sum_{j=a}^{n} \binom{n}{j} (qu'')^j ((1 - q)d'')^{n - j} - \frac{X''}{(1 + r)^n} \sum_{j=a}^{n} \binom{n}{j} G^j (1 - G)^{n - j}$$

and set $A'' = Gu''/r$, $1 - A'' = (1 - G)d''/r$.

This is not yet the value of a call on a futures contract. Indeed, the above still gives the value of a call on the T-bill (or on the cash). To replicate the call on the futures, we need $k = (1 + r_{T,T_2})/(1 + r^*_{T,T_2})$ of these cash options, each with strike price $X'' = X/k$. So a call on a futures contract is worth.

$$Cf_0 = f_0 \sum_{j=a}^{n} \binom{n}{j} A''^j (1 - A'')^{n - j} - \frac{X}{(1 + r)^n} \sum_{j=a}^{n} \binom{n}{j} G^j (1 - G)^{n - j}.$$

ME3. Relate the European put formula,

$$P_0 = \frac{X}{(1 + r)^n} \text{Prob}(j < a; n, q) - \frac{S_0}{(1 + r^*)^n} \text{Prob}(j < a\,; n, \pi)$$

to the European call formula and check for Put Call Parity. Recall that $j$ is the (*ex ante* unknown) number of up-changes, and $a$ the minimum number of "ups" required to bring $S_T$ above $X$.

A3.

$$P_0 = \frac{X}{(1+r)^n}\,[1 - \text{Prob}(j \geq a;n,q)] - \frac{S_0}{(1+r^*)^n}\,[1 - \text{Prob}(j \geq a;n,\pi)]$$

$$= \frac{S_0}{(1+r^*)^n} - V_0 + V_0\,\text{Prob}(j \geq a;n,\pi) - \frac{X}{(1+r)^n}\,\text{Prob}(j \geq a;n,q)$$

$$= \frac{X}{(1+r)^n} - V_0 + C_0.$$

# Chapter 8 Pricing European Options: The Lognormal Model

## Quiz

### Matching Questions

Match each phrase or set of symbols with the statement(s) to which it corresponds most.

(a) Continuous trading
(b) Continous exchange rate process
(c) Lognormality
(d) $\sigma^2$
(e) constant $\sigma^2$
(f) $N(d_i)$
(g) $N(d_1)$
(h) $N(d_2)$
(i) $\Delta$
(j) volatility
(k) $F_{t,T}\, N(d_1)$
(l) $E_T(S_t)\, N(d_1)$
(m) $CEQ_t(\tilde{C}_T)$

________ 1. Risk-adjusted partial mean of $\tilde{S}_T$.
________ 2. The assumption that corresponds to the assumption in the binomial model of changes in the spot exchange rate.
________ 3. An option's exposure to $f_{t,T}$.
________ 4. Cumulative standard normal probability.
________ 5. The assumption which ensures that changes in the exchange rate are small over short intervals.
________ 6. The probability of exercising.
________ 7. The partial mean $\tilde{S}_T$.
________ 8. The assumption that makes it possible to continuously rebalance the portfolio based on the option's ever-changing exposure.
________ 9. The risk-adjusted expectation of the call value at time $T$.
________ 10. The number of currency units needed to replicate an option.
________ 11. The constant variance of the log exchange rate.
________ 12. The condition which says that the continuously compounded changes in the exchange rate are normally distributed.
________ 13. The assumption that corresponds to the assumption in the binomial model of the multiplicativity of the changes in the spot exchange rate.
________ 14. *p.a.* standard deviation (of an exchange rate).
________ 15. Cumulative probability of a call ending in the money.

Ans. 1. k; 2. c; 3. i; 4. f; 5. b ; 6. h; 7. l; 8. a; 9. m; 10. i; 11. d; 12. c; 13. e; 14. j; 15. g.

### Additional Quiz Questions

Q1. Are the following comments true or false. Please explain your answers.
(a) A major flaw of the Black-Scholes model is that it assumes risk neutrality.

(b) The Black-Scholes model cannot be adjusted for interest rate risk.

A1. (a) True. This assumption creeps in when $E_t(\tilde{S}_T)$ is set to $F_{t,T}$, and is used once more when discounting is done at the risk-free rate.
(b) False. Use Merton's or Grabbe's variant.

Q2. Suppose that, everything else held constant, $S$ and $X$ change by the same factor. How does this affect:
(a) The risk-adjusted probability of exercising?
(b) The partial mean of $\tilde{S}_T$ (above $X$)?
(c) The value of a call?
(d) The value of a put?

A2. (a) unaffected; (b), (c), and (d) all change by the same factor as $S$ and $X$.

Q3. Consider the Garman-Kohlhagen call pricing formula :

$$C_t = S\, e^{-r'(T-t)}\, N(d_1) - X\, e^{-r(T-t)}\, N(d_2).$$

(a) $N(d_2)$ refers to what area under the normal curve?
(b) $N(d_2)$ refers to the (risk-adjusted) probability of what?
(c) The size of the domestic deposit held in the replicating portfolio is given by?
(d) The analogous (*foreign*) interest-earning currency held in the replicating portfolio is given by?
(e) The home currency value of the foreign deposit in the replicating portfolio is given by?
(f) The hedge ratio is given by?

A3. (a) The area under the curve to the left of $d_2$.
(b) Of $S>X$, that is exercising.

(c) By $X\, e^{-rt}\, N(d_2)$, or $\dfrac{X}{1 + r_{t,T}}\, N(d_2)$.

(d) By $e^{-r^*t}\, N(d_1)$, or $\dfrac{1}{1 + r^*_{t,T}}\, N(d_1)$.

(e) By $S\, e^{-r^*t}\, N(d_1)$, or $S\, \dfrac{1}{1 + r^*_{t,T}}\, N(d_1)$.

(f) By $e^{-r^*t}\, N(d_1)$, or $\dfrac{1}{1 + r^*_{t,T}}\, N(d_1)$.

Q4. Consider the Garman-Kohlhagen put pricing formula :

$$P = X\, e^{-r(T-t)}\, N(e_1) - S\, e^{-r^*(T-t)}\, N(e_2).$$

(a) $N(e_1)$ refers to what area under the normal curve.
(b) $N(e_1)$ refers to the (risk-adjusted) probability of what.
(c) The size of the domestic deposit held in the replicating portfolio is given by?
(d) The analogous (*foreign*) interest-earning currency held in the replicating portfolio is given by.

(e) The home currency value of the foreign deposit in the replicating portfolio is given by.
(f) The hedge ratio is given by.

A4. (a) The area under the curve to the left of $e_1$.
(b) If $S < X$, that is exercising.

(c) By $X\, e^{-r(T-t)}\, N(e_1)$, or $\frac{X}{1 + r_{t,T}}\, N(e_1)$.

(d) By $e^{-r^*(T-t)}\, N(d_1)$, or $\frac{1}{1 + r^*_{t,T}}\, N(d_1)$.

(e) By $S\, e^{-r^*(T-t)}\, N(e_2)$, or $- S\, \frac{1}{1 + r^*_{t,T}}\, N(e_2)$.

(f) By $e^{-r^*(T-t)}\, N(e_2)$, or $\frac{-\,1}{1 + r^*_{t,T}}\, N(e_2)$.

## Exercises

E1. Suppose the current spot rate for the Timbuktu Dirham is USD/TID 50. Use the Garman-Kohlhagen formula to calculate the price of a European call option on TID, if the strike price is USD/TID 45, the time to maturity of the call is 91 days, the variance of the spot exchange is 20 percent and the USD interest rate is 6 percent, while the TID interest rate is 0 percent.
(a) What is the option's price?
(b) What is the delta of this call option.
(c) What would the price of this option be if it were an American option?

A1. European call price = USD 5.926. The delta is given by $N(d_1)$ = .985, since $\frac{1}{1 + r^*_{t,T}} = 1$.
With a zero foreign rate, an American call will have exactly the same price as the European call, because $C_t^{am} \geq C > S_t = \frac{X}{1 + r_{t,T}} > S_t - X$ = intrinsic value.

E2. Suppose that the USD/CAD spot rate is 0.75 and the volatility of this exchange rate is $\sigma$ = 4 percent *p.a*. The risk-free rate is 7 percent *p.a.* on USD and 10 percent *p.a.* on CAD. Suppose that the exercise price is USD/CAD 0.75 and the European put option matures in 274 days.
(a) Find the value of this option using the Garman-Kohlhagen approach.
(b) What is the put option's exposure?
(c) What would the price of this option be if it were an American option?

A2. European put is worth USD 0.019, with a delta equal to -0.672. As the foreign rate is 3 percent above the domestic rate, the American put price will be virtually the same.

E3. In the call formula, we see $\frac{\ln(S_t/X)}{\sigma_{t,T}}$ inside $N(\bullet)$. This expresses the continuously compounded percentage difference between $S_t$ and $X$ relative to the standard deviation and is a good measure of in-the-moneyness.

(a) Verify that the call price converges to 0 when

$$\frac{\ln(S_t/X)}{\sigma_{t,T}}$$

approaches zero (deep out-of-the-money).

(b) Verify that the call price converges to the forward purchase contract's value when

$$\frac{\ln(S_t/X)}{\sigma_{t,T}}$$

approaches plus infinity (deep in-the-money).

A3. (a) For $S_t/X$ approaching zero, $F_{t,T}/X$ approaches zero too, so $\ln(F_{t,T}/X)$ approaches minus infinity; thus both $N(.)$ factors go to zero.

(b) For $S_t/X$ approaching plus infinity, $F_{t,T}/X$ approaches plus infinity too, so $\ln(F_{t,T}/X)$ approaches plus infinity; thus both $N(.)$ factors approach 1.

E4. Consider a five-year currency option bond which gives the holder the option to cash in the principal either as USD 10,000, or as CHF 20,000. The bond carries a 6 percent USD coupon paid once a year (that is, each annual coupon is USD 600). Assume a flat 9 percent *p.a.* term structure for USD bonds, and a flat 3 percent *p.a.* term structure for CHF bonds. The current exchange rate is $S_t$ = 0.5 USD/CHF, and the volatility is 0.15 (that is, the standard deviation of a yearly continuously compounded exchange rate change is 15 percent). Find the fair price of the bond[5], if you know that the prices of one-, two- ..., five-year calls are 4.36, 6.81, 8.82, 10.54 and 12.01 cents, respectively.

A4. The straight USD bond is worth:

$$10{,}000 - (900 - 600) \times a(9 \text{ percent, five years})$$
$$= 10{,}000 - 300 \times 3.8896512 = \text{USD } 8{,}833.$$

A five-year call on CHF 1 is worth USD 0.1201, and the call bears on CHF 20,000. Thus, the bond's price is 8,833 + (20,000 × 0.1201) = USD 11,235.

E5. Redo Exercise 4, but assume that the option also includes the coupons; the holder can either cash in USD 600, or CHF 1,200, at any coupon date.

A5. You now hold five calls on CHF 1,200 each, in addition to the bond valued in the preceding question. Thus:

$$V_0 = 11{,}235 + (1{,}200 \times \sum_{t=1}^{5} \text{CALL}_t) = 11{,}235 + (1{,}200 \times 0.4254) = 11{,}745.5.$$

E6. Redo Exercise 5, but treat the bond as a CHF 20,000 bond plus five calls on USD at $X'$ = CHF/USD 2. If you sit back and think for two or three minutes, you may be able to answer this question analytically rather than going through the computations as in the previous question.

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5 Hint: fromChapter 10, with a flat 9% term structure, you can price a straight USD bond as $V_0$ = USD 10,000 + (600 - 900) × a($R$,$n$) where $a(R,n) = (1 - (1 + R)^{-n})/R$ = 3.8896512 is the annuity factor for $R$ = 0.09, $n$ = 5, and where USD 900 represents the coupon that would be "normal" under the current circumstances. Therefore, compute the bond's discount as the discounted shortfall in the coupon relative to the normal rate.

A6. Since that bond basically produces the same payoffs, its USD value must be 11,745.5, too. Therefore, looking at everything from a CHF point of view, the bond's CHF value must be $11{,}745.5 \times 2$ = CHF 23,491 if the formula is logical (which it is).

E7. Compute for each of the two preceding exercises how much of the bond's total value is due to the option and how much is due to the straight bond. Explain intuitively why the option component has a much higher relative importance from the USD point of view than from the CHF point of view.

A7. From one angle, the USD straight bond is below par (6 percent is below the going rate of 9 percent), while the CHF straight bond is above par (6 percent is above the going 3 percent rate). Since the total USD value is the same, the options on CHF must be worth a lot in terms of USD, while the options on USD cannot be worth much in terms of CHF.

From a USD perspective, the forward CHF is always above par ($S_t$ = 0.5 USD/CHF), so the risk-adjusted expectations are that the CHF rises by approximately 6 percent/year. This makes the calls valuable. From a CHF point of view, the risk-adjusted expectation is that the USD will depreciate by approximately 6 percent/year. this makes the calls on USD less valuable.

E8. Suppose that we want to lower the USD coupon such that the issue price is around par. Without any calculations, should this crucial coupon rate be:
(a) above 9 percent (the USD rate)?
(b) above 6 percent (the bond's coupon)?
(c) between 6 percent and 3 percent?
(d) below 3 percent (the CHF rate)?
Why?

A8. At 3 percent a straight CHF bond would already be at par, so adding calls on USD would lift the price above par. To bring down the price, lower the coupon to below 3 percent, thus decreasing the present value of both the interest payments and the calls.

E9. How should we set the bond's coupon rate if we want a bond issue price equal to the par value when:
(a) The option bears on the principal only?
(b) The option also bears on each of the coupons?

A9. Let $c$ = coupon. If the option bears on the principal only, the total value is USD 10,000:

$$10{,}000 \;=\; V_0 \;=\; \frac{10{,}000}{1.09^5} \;+\; c\sum_{t=1}^{5}\frac{1}{1.09^t} \;+\; 20{,}000\ \text{CALL}_5$$

$$= \frac{10{,}000}{1.09^5} \;+\; (c \times a(9\%, 5)) \;+\; 20{,}000\ \text{CALL}_5.$$

So,

$$10{,}000\,(1 - 1/1.09^5) - 20{,}000 \times 0.1201 = c \times 3.8896512.$$

Therefore, $c$ = USD 282.5. If the option also applies to the coupons (that is, you choose between USD c or CHF 2c), we impose:

$$10{,}000 = \frac{10{,}000}{1.09^5} + c \sum_{t=1}^{5} \left[\frac{1}{1.09^5} + 2 \times \mathrm{CALL}_t\right] + 20{,}000\ \mathrm{CALL}_5.$$

Then,

$$10{,}000\ (1 - 1/1.09^5) - 20{,}000 \times .1201 = c \times (3.8896512 + 2 \times 0.4254)$$

Therefore, $c$ = USD 232 (or CHF 464 if the option is exercised).

General comment: this last set of exercises was meant to familiarise you with the formula and to make you think about options. For practical computations of this type, your approach should be much more careful. First, you would probably not work with a flat 3 percent or 9 percent five-year bond yield. You would extract "spot" interest rates on $t$-year zero-coupon bonds from available bond data and use a different interest rate for each horizon. (See Chapter 9 for more on this). Second, you would not use the same annualized variance of d$S$/$S$ for all horizons, because for exchange rates, the total variance increases less than proportionally with time. Finally, since over such long horizons, interest rate uncertainty is not trivial and since exchange rate movements are intimately linked to interest rate changes, you should use the Grabbe model (with interest rate risk) rather than the Garman-Kohlhagen model.

## Mind-Expanding Exercises

ME1. (Ito's Lemma, Appendix 8B). Assume that the price level $x$ follows $\frac{dx}{x} = a\ dt + s\ dz$.

Show, from Ito's Lemma, how the mean percentage change in the purchasing power $\Pi = x^{-1}$ is always less, in absolute value, than the mean inflation rate, but that the random components in the inflation rate and in the percentage change of the purchasing power are the same, up to the sign.[6].

A1. From $\Pi_x = -\frac{1}{x^2}$ and $\Pi_{xx} = 2\frac{1}{x^3}$, you obtain:

$$d\Pi = \Pi_x\ dx + \frac{1}{2}\ \Pi_{xx}\, x^2\, \sigma^2\, dt\ ,$$

$$= [-\frac{1}{x^2}]\ dx + \frac{1}{2}\ [2\frac{1}{x^3}]\, x^2\, \sigma^2\, dt = -\frac{1}{x}\frac{dx}{x} + \frac{1}{x}\sigma^2\, dt$$

which implies:

6 Interpret x as the consumption price level, i.e., the price of a standard basket of goods. The purchasing power $\Pi$ of on unit of currency is defined as the number of consumption baskets you can buy with one unit, that is, $\Pi = 1/x$. If there is 10% inflation, x goes up from 100 to 110. The purchasing power decreases, but by less than 10%: $\Pi$ goes from 0.01 to 0.0090909, which represents a drop of 9.0909%. If the inflation rate is smaller, the change in the purchasing power becomes closer to the inflation rate (up to the sign), but it always remains smaller in absolute terms.

$$\frac{d\Pi}{\Pi} = -\frac{dx}{x} + \sigma^2\, dt = -(\pi - \sigma^2)\, dt - \sigma\, dz.$$

In words: if there is randomness in the inflation rate, the mean *p.a.* change in purchasing power is always lower, in absolute value, than the mean *p.a.* inflation rate; but the difference between the random components of inflation and change in purchasing power become negligible over short intervals.

ME2. Derive the value of a call, expiring at $T_1$ and with a strike price $X$, on a futures contract expiring at $T_2$. Ignore interest risk.

A2. Such a call is worth $\frac{1 + r_{T_1,T_2}}{1 + r^*_{T_1,T_2}}$ times the value of a call on the cash with strike price $X'' = X \times \frac{1 + r^*_{T_1,T_2}}{1 + r_{T_1,T_2}}$; so:

$$CF_t = \frac{1 + r_{T_1,T_2}}{1 + r^*_{T_1,T_2}} \left[\frac{S_t}{1 + r^*_{t,T_1}} N(d_1) - \frac{X''}{1 + r_{t,T_1}} N(d_2)\right]$$

$$= \frac{1 + r_{T_1,T_2}}{1 + r^*_{T_1,T_2}} \frac{S_t}{1 + r^*_{t,T_1}} N(d_1) - \frac{X}{1 + r_{t,T_1}} N(d_2).$$

Multiply and divide the first term by $1 + r_{t,T_1}$, and note that, under interest rate certainty, $(1 + r_{t,T_1})(1 + r_{T_1,T_2}) = (1 + r_{t,T_2})$ and similarly for $r^*$; this leads to:

$$Cf_t = \frac{1 + r_{t,T_2}}{1 + r^*_{t,T_2}} \frac{S_t}{1 + r_{t,T_1}} N(d_1) - \frac{X}{1 + r_{t,T_1}} N(d_2)$$

$$= \frac{F_{t,T_2}}{1 + r_{t,T_1}} N(d_1) - \frac{X}{1 + r_{t,T_1}} N(d_2)$$

Note that:

$$d_1 = \frac{\ln \frac{F_{t,T_1}}{X''} + \frac{1}{2}\sigma^2_{t,T_1}}{\sigma_{t,T_1}}$$

In this expression, we again use $X'' = X\frac{1 + r^*_{T_1,T_2}}{1 + r_{T_1,T_2}}$; $F_{t,T_1} = S_t \frac{1 + r_{t,T_1}}{1 + r^*_{t,T_1}}$; $(1 + r_{t,T_1}) \times (1 + r_{T_1,T_2}) = (1 + r_{t,T_2})$; and $F_{t,T_2} = S_t \frac{1 + r_{t,T_2}}{1 + r^*_{t,T_2}}$. This leads to:

$$d_1 = \frac{\ln \frac{F_{t,T_2}}{X} + \frac{1}{2}\sigma^2_{t,T_1}}{\sigma_{t,T_1}}.$$

ME3. A call on *one* DEM at $X$ = 0.5 USD/DEM each is a put on *one-half* USD at $X'$ = DEM/USD 2. Or more generally, a call on DEM 1 at a strike price of $X$ is a put on USD $X$ at a strike price of $X' = 1/X$. Verify this using the continuous-time formula. (Hint: since we are changing the reference currency, it is advisable not to use a $r$, $r^*$ notation. You will be less confused if you use a notation like $R_\$$ (for USD) and $R_M$ (for DEM)).

A3. The call on DEM $\frac{1}{X}$ is priced in USD. Its DEM value therefore is:

$$P_{t,\text{DEM}} = \frac{1}{X} C_{t,\text{USD}}/S_t = \frac{1/X}{1 + r_{M,t,T}} N(d_1) - \frac{1/S_t}{1 + r_{\$,t,T}} N(d_2)$$

or

$$P_{t,\text{DEM}} = \frac{X'}{1 + r_{M,t,T}} N(d_1) - \frac{S'_t}{1 + r_{\$,t,T}} N(d_2) ,$$

where $X' = 1/X$ is the strike price from the DEM point of view, and $S'_t = 1/S_t$ is the exchange rate from the DEM point of view. Finally, note that:

$$d_1 = [\ln \frac{F_{t,T}}{X} + \frac{1}{2} \sigma^2_{t,T}] / \sigma_{t,T}$$

$$= [\ln \frac{1/X}{1/F_{t,T}} + \frac{1}{2} \sigma^2_{t,T}] / \sigma_{t,T}$$

$$= [\ln \frac{X'}{F'_{t,T}} + \frac{1}{2} \sigma^2_{t,T}] / \sigma_{t,T}$$

$$= [- \ln \frac{F'_{t,T}}{X'} + \frac{1}{2} \sigma^2_{t,T}] / \sigma_{t,T} = -d_2$$

$= e_1$ in the put formula of Section 4.

So $d_1$ in the (USD-denominated) call formula corresponds to $e_1$ in the (DEM-denominated) put formula. By similar reasoning, $d_2$ in the USD call formula corresponds to $e_2$ in the DEM put formula.